

## Lesson 14. Improving Search: Convexity and Optimality

### 1 Overview

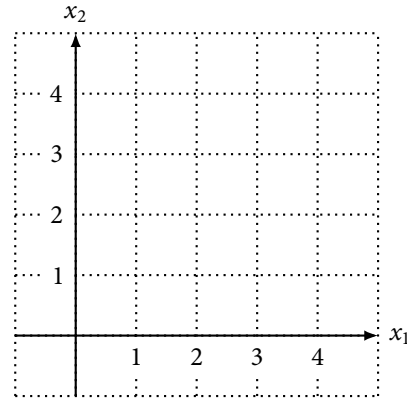
- 1: Find an initial feasible solution  $\mathbf{x}^0$
- 2: Set  $k = 0$
- 3: **while**  $\mathbf{x}^k$  is not locally optimal **do**
- 4:     Determine a new feasible solution  $\mathbf{x}^{k+1}$  that improves the objective value at  $\mathbf{x}^k$
- 5:     Set  $k = k + 1$
- 6: **end while**

- Step 3 – Improving search converges to local optimal solutions, which aren’t necessarily globally optimal
- Wishful thinking: when are all local optimal solutions are in fact globally optimal?

### 2 Convex sets

**Example 1.** Let  $\mathbf{x} = (1, 1)$  and  $\mathbf{y} = (4, 3)$ . Compute and plot  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$  for  $\lambda \in \{0, 1/3, 2/3, 1\}$ .

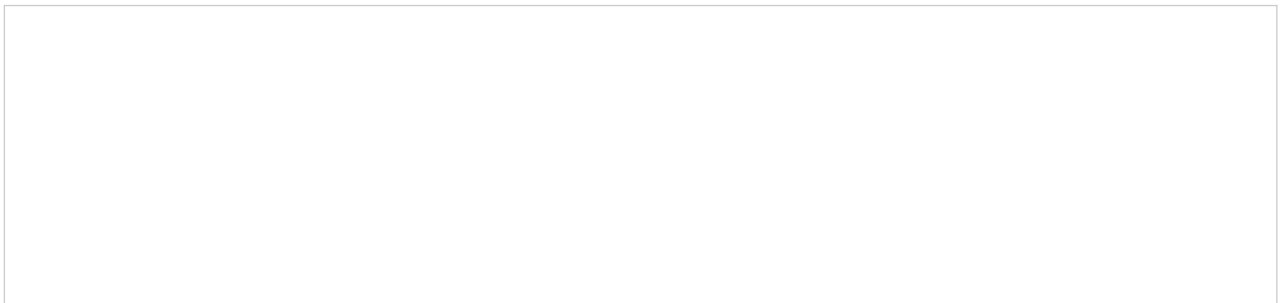
$\lambda$	$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$
0	
1/3	
2/3	
1	



- Given two solutions  $\mathbf{x}$  and  $\mathbf{y}$ , the **line segment** joining them is

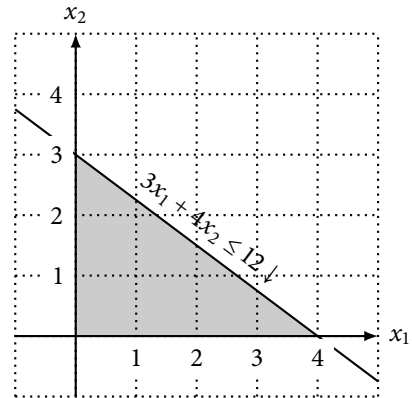
$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \quad \text{for } \lambda \in [0, 1]$$

- A feasible region  $S$  is **convex** if for all  $\mathbf{x}, \mathbf{y} \in S$ , then  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S$  for all  $\lambda \in [0, 1]$ 
  - A feasible region is convex if for any two solutions in the region, all solutions on the line segment joining these solutions are also in the region
- Graphically: convex vs. nonconvex



**Example 2.** Show that the feasible region of the LP below is convex.

$$\begin{aligned} \text{minimize} \quad & 3x_1 + x_2 \\ \text{subject to} \quad & 3x_1 + 4x_2 \leq 12 \quad (1) \\ & x_1 \geq 0 \quad (2) \\ & x_2 \geq 0 \quad (3) \end{aligned}$$



*Proof.*

- Let  $\mathbf{x} = (x_1, x_2)$  and  $\mathbf{y} = (y_1, y_2)$  be arbitrary points in the feasible region
- In other words,  $\mathbf{x}$  and  $\mathbf{y}$  satisfy (1), (2), (3)
- We need to show  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$  also satisfies (1), (2), (3) for any  $\lambda \in [0, 1]$
- Note that

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} =$$

- One constraint at a time: does  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$  satisfy (1)?

$$3(\lambda x_1 + (1 - \lambda)y_1) + 4(\lambda x_2 + (1 - \lambda)y_2) = \lambda(3x_1 + 4x_2) + (1 - \lambda)(3y_1 + 4y_2)$$

$$\leq 12\lambda + 12(1 - \lambda)$$

$$= 12$$

- We can show  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$  also satisfies (2) and (3) in a similar fashion □

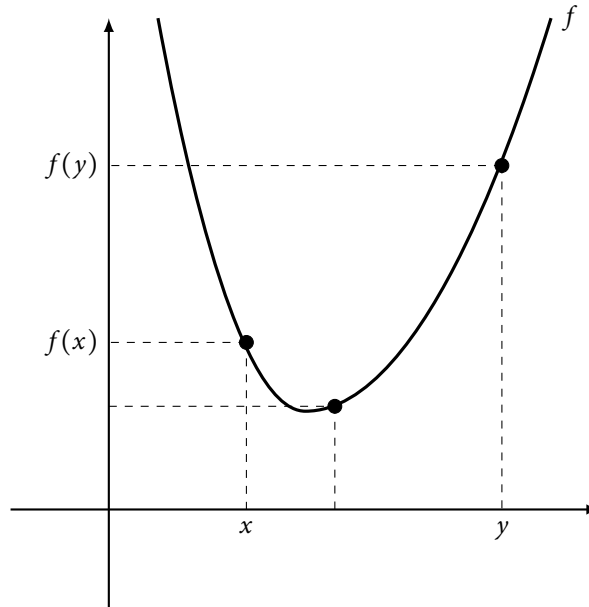
- **In general, the feasible region of an LP is convex**

### 3 Convex functions

- Given a convex feasible region  $S$ , a function  $f(\mathbf{x})$  is **convex** if for all solutions  $\mathbf{x}, \mathbf{y} \in S$  and for all  $\lambda \in [0, 1]$

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$$

- Graphically:



**Example 3.** Show that the objective function of the LP in Example 2 is convex.

*Proof.*

- Let  $f(\mathbf{x}) = 3x_1 + x_2$
- For any  $\mathbf{x}$  and  $\mathbf{y}$ , we have:

$$\begin{aligned} f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) &= 3(\lambda x_1 + (1 - \lambda)y_1) + (\lambda x_2 + (1 - \lambda)y_2) \\ &= \lambda(3x_1 + x_2) + (1 - \lambda)(3y_1 + y_2) \end{aligned}$$

$$= \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) \quad \square$$

- In general, the objective function of an LP – a linear function – is convex**

## 4 Minimizing convex functions over convex sets

**Big Theorem.** Consider the following optimization model:

$$\begin{aligned} & \text{minimize} && f(\mathbf{x}) \\ & \text{subject to} && g_i(\mathbf{x}) \begin{cases} \leq \\ \geq \\ = \end{cases} b_i \quad \text{for } i \in \{1, \dots, m\} \end{aligned} \quad (*)$$

Suppose  $f$  is convex and the feasible region is convex. If  $\mathbf{x}$  is a local optimal solution, then  $\mathbf{x}$  is a global optimal solution.

*Proof.*

- By contradiction – suppose  $\mathbf{x}$  is not a global optimal solution
- Then there must be another feasible solution  $\mathbf{y}$  such that  $f(\mathbf{y}) < f(\mathbf{x})$
- Take  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$  really close to  $\mathbf{x}$  ( $\lambda$  really close to 1)
- Since the feasible region is convex,  $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$  is also in the feasible region
- We have that:

$$\begin{aligned} f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) &\leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) && \text{(since } f \text{ is convex)} \\ &< \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{x}) && \text{(since } f(\mathbf{y}) < f(\mathbf{x})\text{)} \\ &= f(\mathbf{x}) \end{aligned}$$

- Therefore:  $f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) < f(\mathbf{x})$
- $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$  is a feasible solution in the neighborhood of  $\mathbf{x}$  with better objective value than  $\mathbf{x}$
- This contradicts  $\mathbf{x}$  being a local optimal solution!
- Therefore,  $\mathbf{x}$  must be a global optimal solution □

- Remember that an improving search algorithm finds local optimal solutions
- Since the objective function of an LP is convex, and the feasible region of an LP is convex:

**Big Corollary 1.** A global optimal solution of a minimizing linear program can be found with an improving search algorithm.

- A similar theorem and corollary exists when maximizing concave functions over convex sets
  - See pages 222–225 in Rader for details

**Big Corollary 2.** A global optimal solution of a maximizing linear program can be found with an improving search algorithm.